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Note

Estimates of the Orthogonal Polynomials with Weight $\exp(-x^m)$, m an Even Positive Integer

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INTRODUCTION

Let m be an even positive integer and let $w(x)=\exp(-x^m)$. The orthogonal polynomials associated with $w(x)$, denoted by $\{p_n(x)\}_{n=0}^{+\infty}$, are defined by

$$\int_{-\infty}^{+\infty} p_n(x)\, p_k(x)\, w(x)\, dx = 0, \qquad n \neq k, \\ = 1, \qquad n = k \tag{1}$$

and $p_n(x)=\gamma_n x^n + \cdots$, with $\gamma_n > 0$.

A survey by Nevai [9] contains recent investigations into the properties of this class of orthogonal polynomials.

The zeros of $p_n(x)$ are all real and distinct and are denoted by

$$x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

The main result of this paper is the following theorem.

THEOREM 1. *For $n = 1, 2,\ldots$ the following estimates hold*:

(i) $\quad p_n^2(x)\exp(-x^m) \leqslant a/\sqrt{x_{ln}^2 - x^2} \qquad$ when $\quad |x| \leqslant x_{1n}$,

for some fixed positive number a;

(ii) $$\max_{x\in\mathbb{R}} p_n^2(x)\exp(-x^m) = b_n(n^{1/3-1/m}),$$

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for some sequence $\{b_n\}_{n=1}^{\infty}$ *of positive numbers satisfying* $0 < \underline{\lim}_{n\to\infty} b_n \leqslant \overline{\lim}_{n\to\infty} b_n < \infty$.

For $m=2$ these results are known: (i) is due to Erdélyi [1]; (ii) comes from Plancheral–Rotach asymptotics [11, p. 201] and Sonin's theorem [11, p. 166]. When $m \geqslant 4$ these results are new although part (i) contains a result of Nevai's [8] when $|x| \leqslant (1-\varepsilon)\, x_{1n}$ for any $\varepsilon > 0$, and improves an estimate of Lubinsky [3]. Part (ii), when $m \geqslant 4$, disproves a conjecture of Nevai [6] that the sequence

$$M_n = \max_{x\in\mathbb{R}} p_n^2(x)\exp(-x^m), \qquad n = 1, 2, \ldots,$$

is bounded.

The Differential Equation

In order to prove Theorem 1, a differential equation associated with $p_n(x)\exp(-x^m/2)$ is obtained. This has been done when $m=2$, 4, and 6 (see [11, 7, and 10], respectively).

THEOREM 2. *For* $n = 1, 2, \ldots$, *let*

$$A_n(x) = m \sum_{i=0}^{(m-2)/2} \binom{2i}{i} \left[\frac{\gamma_{n-1}}{\gamma_n} + c_n(n^{-1+1/m})\right]^{2i+1} x^{m-2i+2},$$

when the real sequence $\{c_n\}_{n=1}^{\infty}$ *is bounded. A differential equation associated with* $p_n(x)$ *is*

$$z'' + \phi_n(x)\, z = 0$$

with

$$z = p_n(x)[\exp(-(x^m/2) + g_n(x)]/A_n^{1/2}(x)$$

where the function $g_n(x)$ *is twice differentiable and uniformly bounded in* n *when* $\varepsilon|x| \leqslant n^{1/m}$ *for* $\varepsilon > 0$, *and*

$$\phi_n(x) = A_n^2(x)\left(1 - \left(\frac{x}{x_{1n}}\right)^2\right) + h_n(x)(n^{1-2/m})$$

where the function $h_n(x)$ *is uniformly bounded in* n *when* $\varepsilon|x| \leqslant n^{1/m}$ *for* $\varepsilon > 0$.

To verify the differential equation, we need not only the proof of Freud's conjecture by Magnus [4] but also the estimate [5]

$$\frac{\gamma_{n-1}}{\gamma_n} = \beta n^{1/m} + d_n n^{-(2m+1)/m}$$

where

$$\beta = \left[\binom{m-1}{m/2} m\right]^{-1/m}$$

and $\{d_n\}_{n=1}^{\infty}$ is a bounded real sequence.

The proof of Theorem 1 is complicated but uses only elementary properties of the differential equation. Theorem 1 has applications to Lagrange interpolation at the zeros of $p_n(x)$ (see, e.g., [2]).

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